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Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability

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Abstract. Two 2D quasiperiodic tilings with generalized tenfold symmetry are derived from the lattice A_4^R, the reciprocal of the root lattice A_4. Both tilings are built from four tiles, triangles in one case, rhombi and hexagons in the other. After a brief description of the tilings and their structures, we introduce the equivalence concept of mutual local derivability. We present its key properties and its application to several tenfold tilings and discuss some implications on a future classification of tilings in position space.

1. Introduction

The discovery of quasicrystals by Shechtman *et al* [1] and the independent examination of certain aperiodic tilings [2-4] has turned the regime between the crystalline and amorphous phases into an active field of solid state research. Certainly, the quasicrystalline structures represent only one additional class of ordered structures which, however, come closest to the crystalline phase. In the light of the classification of the space groups for crystals it is not surprising that a similar classification for quasicrystals has been a challenge ever since their first appearance in experiment.

However, things are more complicated with quasicrystals. Usually one attempts to describe their microscopic spatial arrangement by projection of higher-dimensional periodic structures. On the one hand, this gives rise to infinitely many possibilities of quasiperiodic patterns even with the same (generalized) symmetry which does not look like the solution of the classification problem. On the other hand, the Fourier modules—i.e., the sets of points in the reciprocal space on which the Fourier transform of, e.g., delta scatterers on special positions of the patterns (like vertices of tilings) may have non-vanishing amplitudes—might be equal (up to similarity transformations) for different projection models. Therefore, a first classification of quasiperiodic patterns is achieved by the classification of the possible modules with a certain, especially non-crystallographic, symmetry. This has been done for the most important set of modules [5].

But, from the point of view of tilings, this classification seems too coarsely grained. In particular, there is no obvious transition scheme from one pattern to another if they only share the Fourier module and, perhaps, the generalized point symmetry. And

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even if there is one in special examples, it does not follow from intrinsic properties of the Fourier module. Furthermore, we would like to emphasize that such a transition rule should be local and should make no substantial reference to an embedding in higher-dimensional spaces. (This does not exclude that a given embedding might facilitate the finding of a local transition rule.)

Some remarks concerning the physical significance of tilings are in order. A characteristic feature of the tilings under consideration is a certain type of pattern repetition (which will be defined more precisely in section 3 yielding the term '(w, t)-repetitivity'). Not every material which exhibits sharp diffraction peaks can necessarily be described as a decoration of such a repetitive tiling. For an abstract example one may take a continuous quasiperiodic function (in the sense of Bohr [6]) which will never fit into the tiling scheme. Stability calculations including relaxations on perfect tilings [7] seem to point to this direction. On the other hand, at least in some cases, the experimental situation supports the tiling hypothesis: one can mention the STM results of Kortan *et al* [8] as well as direct structure determination for AlCuFe [9] where the atomic hypersurfaces derived from the Patterson analysis are amazingly flat in hyperspace, which means that repetitivity of the set of positions of atoms is fulfilled in good approximation; also the tiling overlay to decagonal AlMnPd [10] looks very convincing. If one accepts this interpretation of the data, one should take into consideration the following fact. Given a point set, (w, t)-repetitive as defined in section 3, that is locally finite, one can always find a (w, t)-repetitive tiling with a finite set of fundamental tiles so that the points can be seen as a decoration of the tiles. (To prove this statement, one would just take the Delaunay complex [11] of the point set. The finiteness of the set of prototiles then follows from the (w, t)-repetitivity and local finiteness of the resulting tiling.) Transcribed into the physical context, this means that a (w, t)-repetitive arrangement of atomic positions that stems from flat hypersurfaces *can always be seen as a decoration of a suitable (w, t)-repetitive tiling.*

Nevertheless, we tend to the opinion that the relevance of tilings is by no means completely settled. To attack this problem one certainly needs additional concepts in position space because many aspects of tilings are defined there and possess only an extremely complicated counterpart in Fourier space. In the present paper, we will investigate when two different tilings represent the same long range translational order and can therefore be regarded equivalent in this respect. This does not solve the classification problem, because we disregard orientational symmetry, but may be a first step towards a solution. We start in an illustrative fashion with some tenfold tilings (however, not from the generalized Penrose tilings [12, 13]) and apply the concepts there. Our examples will lead us to the conclusion that an effective classification of tilings should be less crude than that of the Fourier modules [5].

The paper is organized as follows. In section 2, we present two quasiperiodic tilings with generalized tenfold symmetry as derived from the 4D lattice A_4^{R} which is the reciprocal of the so-called root lattice A_4 [14, 15]. Here, we can be brief because the situation closely resembles that of A_4 where the Penrose tiling [2] and the so-called triangle pattern ($\mathcal{T}^*_{A_4}$) [15, 16] have been derived. We thus explicitly have four different patterns at our disposal which do in fact share the same Fourier module (after an appropriate similarity transformation).

We take this situation as the starting point for more general considerations in section 3, where we propose 'mutual local derivability' as a reasonable equivalence concept with respect to the description of quasicrystals. We give the key properties of this concept and illustrate it by application to the tenfold patterns mentioned above.

Furthermore, a necessary and sufficient condition for mutual local derivability of patterns which are generated by a projection method is derived (leaving some details to the appendix). This is followed by some concluding remarks on a possible meaning of the resulting equivalence classes for a future classification of quasiperiodic patterns.

2. Derivation of tenfold symmetric tilings with four different tiles from the lattice A_4^R

The geometric frame and the notation in the case of A_4^R is (almost) identical to the treatment of the so-called root lattice A_4 [15], wherefore we can be brief. If $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_5$ denote the standard basis vectors of 5D Euclidean space $\mathbb{R}^5$, we can write the 4D lattice A_4^R as the integer linear combinations of the vectors

$$\boldsymbol{a}_i = \boldsymbol{e}_i - \tfrac{1}{5}s \qquad i = 1, \ldots, 4 \qquad s = \boldsymbol{e}_1 + \ldots + \boldsymbol{e}_5.$$

For convenience, we define $\boldsymbol{a}_5$ as well, but one finds $\boldsymbol{a}_1 + \ldots + \boldsymbol{a}_5 = 0$. The lattice A_4^R is confined to the 4D subspace

$$S = \{x \in \mathbb{R}^5 \mid x \cdot s = 0\}$$

and possesses the point group $H = S_5 \times Z_2$, which is naturally the same as that of the root lattice A_4, the latter consisting of all integer linear combinations of the four vectors $\boldsymbol{e}_1 - \boldsymbol{e}_2, \ldots, \boldsymbol{e}_4 - \boldsymbol{e}_5$. (The lattices A_4 and A_4^R are reciprocal to each other, for a more detailed description of their geometry we refer to [14].)

The selection of the 'physical' space is done precisely the same way as in the treatment of A_4 [15]. In fact, one finds

$$S = \mathbb{E}_\parallel \oplus \mathbb{E}_\perp$$

where $\mathbb{E}_\parallel$ and $\mathbb{E}_\perp$ are the invariant subspaces of a subgroup of the point group of A_4^R isomorphic to the dihedral group D_{10}. We shall denote the orthogonal projections onto the spaces $\mathbb{E}_\parallel$ and $\mathbb{E}_\perp$ by the symbols $\pi_\parallel$ and $\pi_\perp$, respectively.

The lattice A_4^R is invariant under the linear transformation defined by

$$\boldsymbol{a}_1 \mapsto \boldsymbol{a}_2 + \boldsymbol{a}_5 \qquad (+\text{cyclic permutation of indices}) \tag{1}$$

which is an element of the unimodular group attached to A_4^R. It turns out that this transformation acts as a contraction in $\mathbb{E}_\parallel$ (by a factor $1/\tau$) and a dilation in $\mathbb{E}_\perp$ (by $-\tau$). Therefore, as is well known from other examples, this contraction (leaving the Fourier module invariant) forms a candidate for a deflation transformation of the quasiperiodic tilings to be derived. However, as will turn out in a shortwhile, the existence of a deflation transformation does not automatically result from the existence of a higher-dimensional lattice transformation compatible with $\mathbb{E}_\parallel$ and $\mathbb{E}_\perp$, but additionally depends on the specific details of the construction of the tiling.

For the very construction we now need the Voronoi domain $V(0)$ around the origin, its 2D boundaries (or 2-boundaries for short), and the corresponding dual 2-boundaries of the Delaunay cells of A_4^R [17]. The Voronoi domain $V(0)$ is the so-called permutohedron [14],

$$V(0) = \operatorname{conv}\{\sigma(P) \mid \sigma \in S_5, P = \tfrac{1}{5}(-2, -1, 0, 1, 2)\} \tag{2}$$

a 4D polytope the vertices of which are deep holes (or interstitials) of the lattice A_4^R with distance $\sqrt{2/5}$ to the nearest lattice points. The 3-boundaries are 10 truncated octahedra (of the form of bcc Wigner-Seitz cells) and 20 hexagonal prisms, the dual objects being the line segments generated by the Voronoi vectors $\pm\boldsymbol{a}_i$ $(1 \leqslant i \leqslant 5)$, $\pm(\boldsymbol{a}_i + \boldsymbol{a}_j)$ $(1 \leqslant i < j \leqslant 5)$, respectively. The 2-boundaries are regular squares and hexagons while the dual 2-boundaries are regular triangles. The two kinds of 2-boundaries split into four classes each that will correspond to the four different tiles of the tilings later on. The 1-boundaries are 240 edges of equal length $\sqrt{2}/5$, the dual 3-boundaries of which are two types of non-regular simplices. Finally, the 0-boundaries are just the 120 vertices displayed in equation (2). Their dual 4-boundaries (the Delaunay cells of A_4^R) are non-regular 4-simplices on a single point group orbit of A_4^R.

The construction of the tilings takes advantage of the one-to-one correspondence between 2-boundaries P of Voronoi cells and 2-boundaries P^* of Delaunay cells. Two different ones are possible, the tiles of which are either projection images of 2-boundaries of the Delaunay complex—yielding a tiling built up from triangles, or the projection images of 2-boundaries of the Voronoi complex—yielding a tiling with rhombi and hexagons. In the first case, taking some point $c \in \mathbb{E}_\perp$, the corresponding tiling consists of all $\pi_\parallel(P^*)$ such that c falls in $\pi_\perp(P)$, in the second case the role of P and P^* is interchanged. (One has to take care that c does not lie on the boundaries of these 'windows' $\pi_\perp(P)$, $\pi_\perp(P^*)$, respectively.) For the general construction method which is used for the definition of the tilings here, we refer to [17], the explicit projection algorithm is described in [15]. Let us anticipate that here we deal with special cases of window mappings as defined in the next section.

We start with the triangular tiling, symbolized by $\mathcal{T}^*_{A_4^R}$, obtained from the Delaunay complex. Here, we find four different tiles, two acute and two obtuse triangles. They combine to 20 possible vertex configurations within the tiling which are given in figure 1 together with their relative frequencies. Each vertex configuration corresponds to an elementary polygon in the projection $\pi_\perp(V(0))$, see figure 2, the area of which provides the value of the relative frequency (for details, see [15]). Figure 3 shows a generic tiling, obtained via the projection algorithm mentioned. (This tiling has been discovered earlier by McMullan [18].)

The other natural possibility to obtain a tiling of $\mathbb{E}_\parallel$ is the projection of certain 2-boundaries of the Voronoi complex. In the resulting tiling, symbolized by $\mathcal{T}_{A_4^R}$, one obtains two different hexagons, a thin and a thick one, and two rhombi, again thin and thick. The latter are similar to the well known Penrose rhombi, but their edge lengths have ratio τ ($:= (1+\sqrt{5})/2$, the golden mean). From the analysis of the Delaunay cells, we have to distinguish four classes of vertex configurations—hence, in a primitive vertex decoration, four different atoms would be possible. All in all, we obtain 13 vertex configurations which are shown in figure 4 together with their relative frequencies. This local analysis stems from the projection images of the Delaunay cells into $\mathbb{E}_\perp$ which are also shown in figure 4. We will refer to the four classes of vertex configurations simply by α, β, γ, and δ, according to figure 4. A part of a $\mathcal{T}_{A_4^R}$-tiling with global fivefold symmetry is displayed in figure 5 (the generalized point symmetry, however, is D_{10} because D_{10}-images are locally isomorphic).

In order to obtain a local description of the $\mathcal{T}_{A_4^R}$ and their deflation/inflation symmetry, it is convenient to pass from the simple tilings to (mathematically) decorated tilings assigning to each vertex its class (α, β, γ, or δ). It is obvious from figure 4 that the decoration of a vertex is completely determined by the set of surrounding tiles, i.e., locally.

A local rule for the deflation of a decorated $\mathcal{T}_{A_4^R}$ can be worked out by an inspection of the 'windows' of the tiles in perpendicular space and their images under the transformation (1). This local rules results in the prescription illustrated in figure 6. It is also possible to perform the reverse inflation of an arbitrary (decorated) $\mathcal{T}_{A_4^R}$-tiling in a completely local way: the hexagons and large rhombi have to replace the configurations shown in figure 6 in the way indicated there; the small rhombi then have to fill

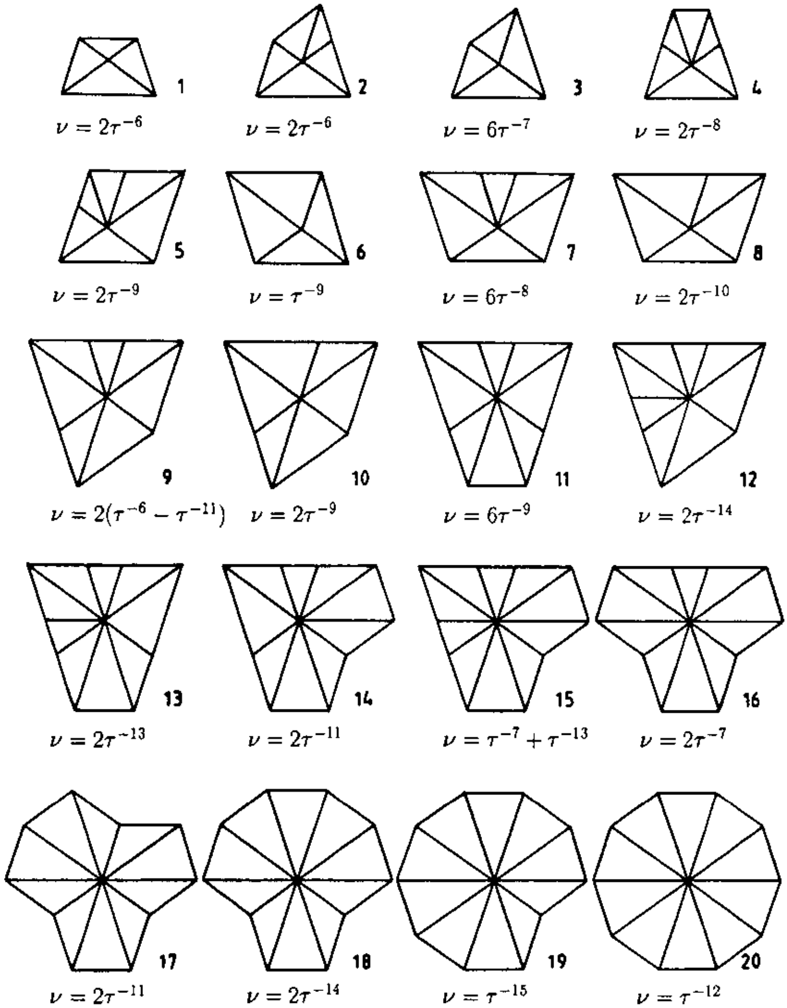


Figure 1. The 20 Vertex configurations of $\mathcal{T}^{*}_{A_4^R}$ with relative frequencies. The numbering refers to figure 2, the contributions of different orientations within the pattern have been summed up for the frequencies.

the remaining gaps. Some tedious, but elementary combinatorics then lead to the following statement: given a tiling of the plane built from the rhombi and hexagons which constitute the $\mathcal{T}_{A_4^R}$, so that only the thirteen vertex configurations $\alpha_1 - \delta_4$ occur in this tiling, then it is always possible to apply the above local rules for inflation as well as for deflation, and, furthermore, the result will be a tiling which also contains only the vertex configurations $\alpha_1 - \delta_4$. By standard arguments one can now conclude that every two tilings which contain only the vertex configurations mentioned, scaled and oriented in the right way, must be locally isomorphic (in the sense defined below). If we now decorate the tiles according to figure 6 and demand that they are glued together face-to-face without mismatch of vertex decorations we find that the vertex configurations $\alpha_1 - \delta_4$ are the only possible ones in an infinite tiling. Consequently, they completely determine the local isomorphism class of the $\mathcal{T}_{A_4^R}$ tiling and may be considered as appropriate 'matching rules'.

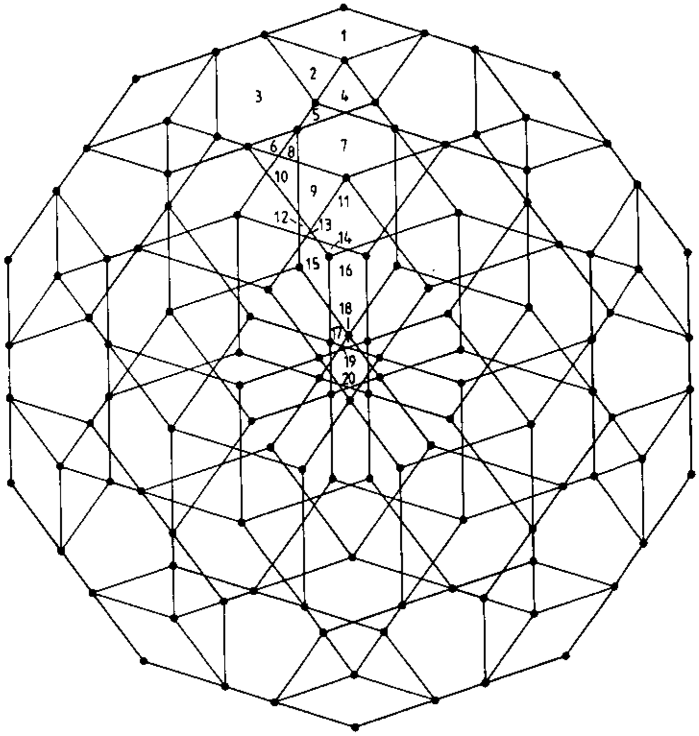


Figure 2. Projection of the Voronoi cell $V(0)$ of the lattice A_4^R into $\mathbb{E}_\perp$. Vertices of $V(0)$ are indicated by small circles. The numbering (1–20) of the elementary polygons refer to the 20 possible vertex configurations in figure 1.

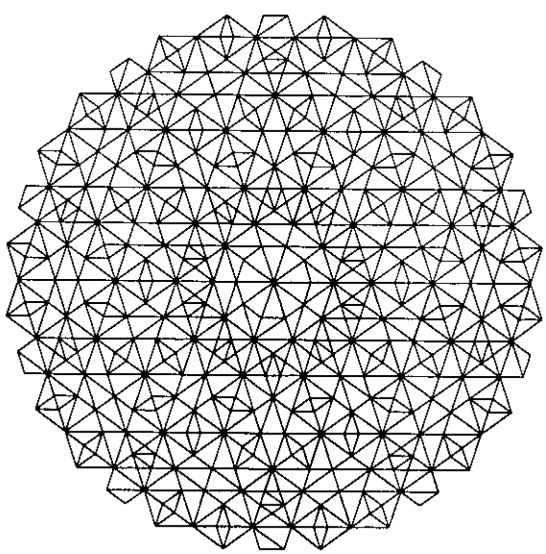


Figure 3. A generic version of the pattern $\mathcal{T}^*_{A_4^R}$. The corresponding local isomorphism (LI-) class does not contain any pattern with exact global five- or tenfold symmetry.

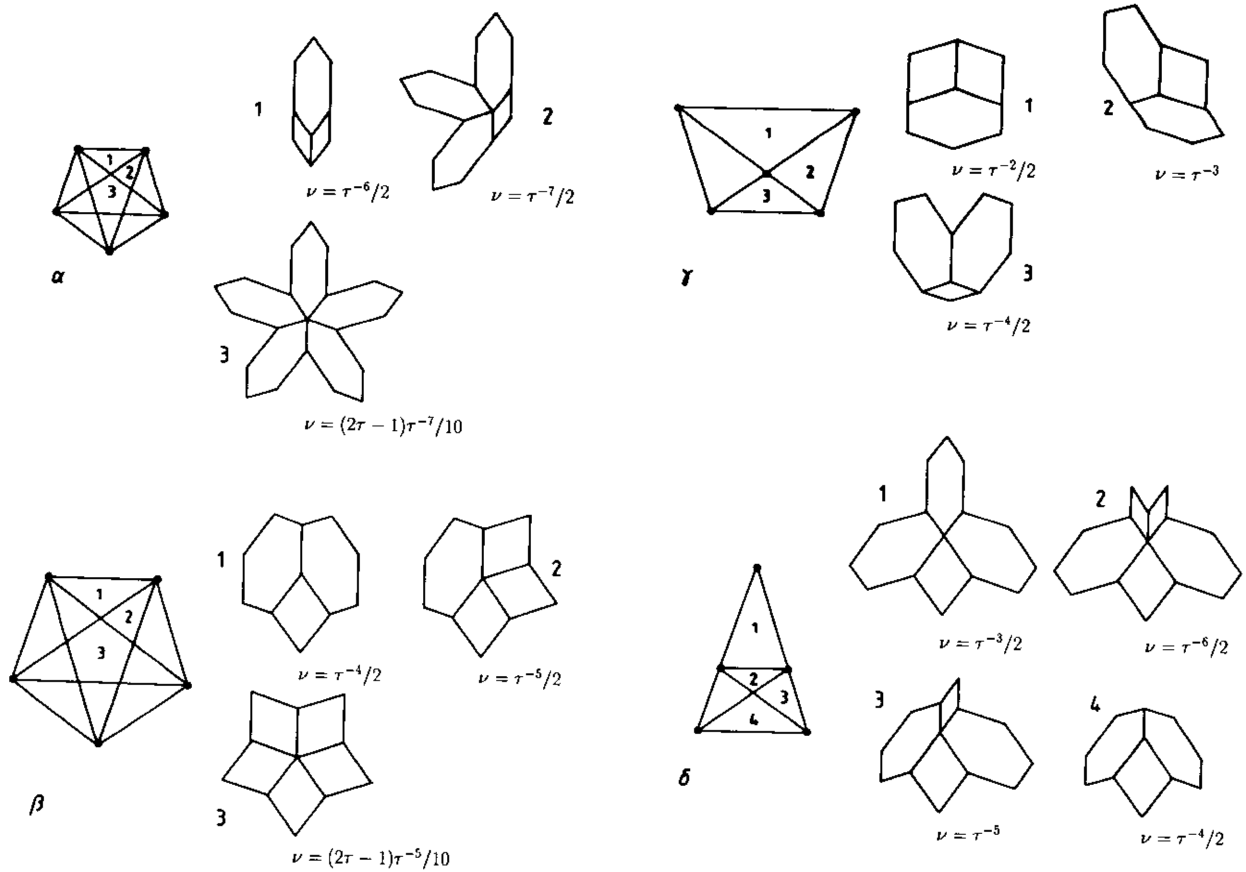


Figure 4. Projection images of representative Delaunay cells ($\alpha - \delta$) into $\mathbb{E}_\perp$ together with the corresponding vertex configurations of the pattern $\mathcal{T}_{A_4^R}$ and their relative frequencies. Again, contributions from different orientations (d_{10} orbits) have been summed up.

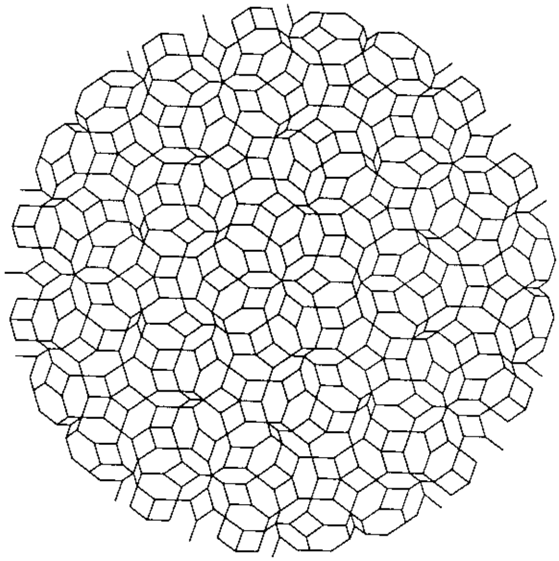


Figure 5. The pattern $\mathcal{T}_{A_4^R}$. We have chosen an exact global fivefold symmetric representative of the LI-class.

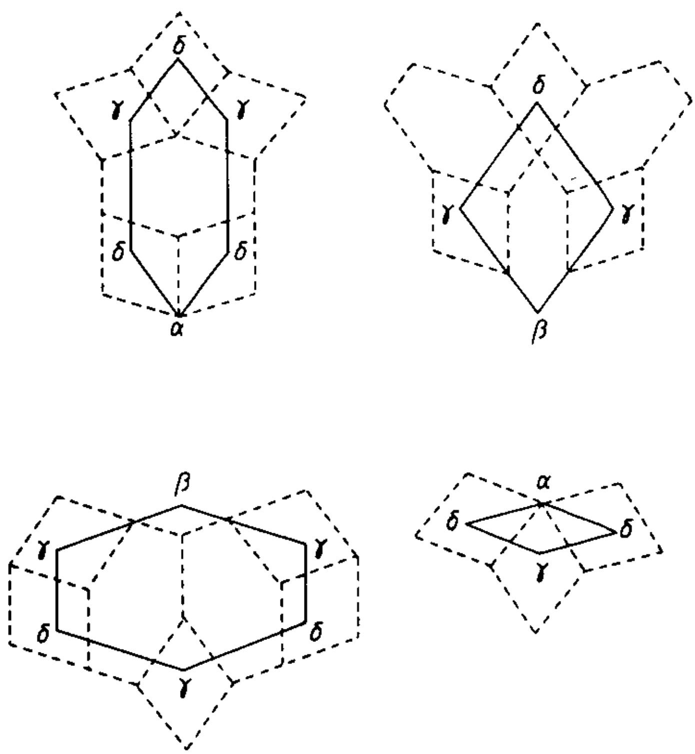


Figure 6. Local deflation of the tiles of $\mathcal{T}_{A_4^R}$. The vertex types are indicated by $(\alpha - \delta)$, the labelling of the deflated tiles can be reconstructed by means of the vertex configurations in figure 4.

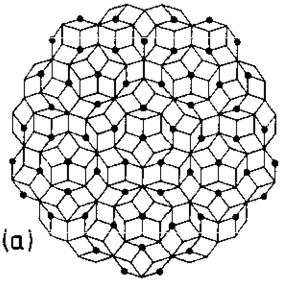
In an earlier publication [15] we analysed the root lattice A_4 in detail and identified the two canonical tilings as the Penrose tiling [2] and the 'triangle pattern' $\mathcal{T}^*_{A_4}$ [16], respectively. Now, since the modules $\pi_\parallel(A_4)$ and $\pi_\parallel(A_4^{\mathrm{R}})$ are identical up to a similarity transformation, we have four different patterns (at first sight), but with the same Fourier module. This situation at least suggests a local analysis [12] of possible relationships between the patterns to which we will turn in the following section.

3. The equivalence concept of mutual local derivability

Now we come to the comparison between different tilings with the same generalized tenfold symmetry. We take four different local isomorphism (LI-) classes of tilings into consideration: the two described in section 2 ($\mathcal{T}_{A_4^{\mathrm{R}}}$, $\mathcal{T}^*_{A_4^{\mathrm{R}}}$), the classical Penrose tiling (see figure 7(a)) (in the version with two rhombi), symbolized $\mathcal{T}_{A_4}$, and a fourth tiling with tenfold symmetry, $\mathcal{T}^*_{A_4}$ (figure 7b), which has been investigated in detail in [15] ('triangle pattern').

Let us first focus our attention on the Penrose tiling on the one hand, and on $\mathcal{T}_{A_4^{\mathrm{R}}}$ on the other. By means of the well known matching rules which determine the Penrose LI-class it is seen that it is always possible consistently to replace the rhombi in a given Penrose tiling according to figure 8. One can see by means of the matching rules given above that the result will be a tiling of the $\mathcal{T}_{A_4^{\mathrm{R}}}$ LI-class. More involved but also elementary is the demonstration of the fact that one will end up with a Penrose tiling, if one performs the inverse replacement in an arbitrary $\mathcal{T}_{A_4^{\mathrm{R}}}$. In comparison with the recipe how to pass, e.g., from the Penrose tiling in the version with rhombi to that

(a)



(b)

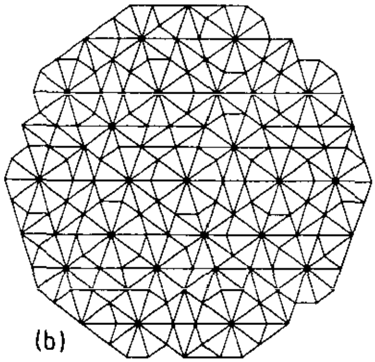


Figure 7. The Penrose pattern $\mathcal{T}_{A_4}$ (a) with two types of vertices and the ('triangle') pattern $\mathcal{T}^*_{A_4}$ (b) as derived from the root lattice A_4.

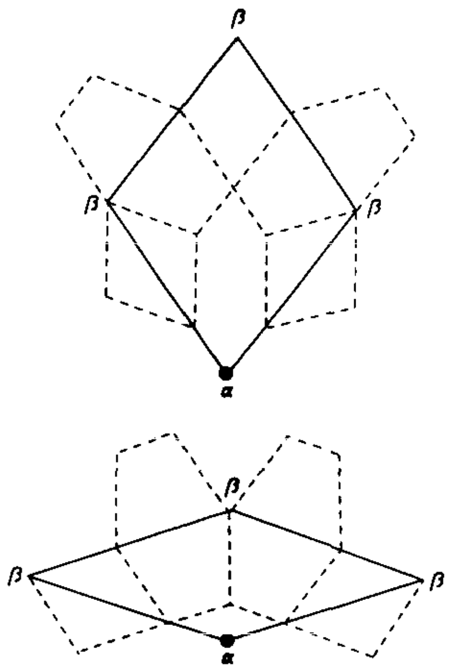


Figure 8. Local transition between the Penrose pattern ($\mathscr{T}_{A_4}$) and the rhomb-hexagon pattern $\mathscr{T}_{A_4^R}$. The vertices of the Penrose pattern are just the vertices of types α and β in $\mathscr{T}_{A_4^R}$.

built up by the so-called kites and darts [2], the procedure is only a little more complicated, and one would tend to the opinion that the Penrose tiling and the tiling $\mathscr{T}_{A_4^R}$ should be considered equivalent with the same right as the two versions of the Penrose tiling itself. Indeed, as a consequence of the above result, both tilings, in a certain sense, contain the same information locally, each is 'locally derivable' from the other.

Because of this possibility of a local transformation of a Penrose tiling into kite-and-dart or rhomb-hexagon ($\mathscr{T}_{A_4^R}$) tilings and vice versa, we conclude: if there is some quasicrystalline material whose microscopic structure could be considered as sort of a decoration of the Penrose tiling, then the kite-and-dart or the rhomb-hexagon tiling would serve as well for this purpose, and vice versa. This resembles the situation in periodic structures, where one also has a lot of possible cell decompositions which can carry the given structure as a decoration. Of course, in the concrete case, there may be one or another cell decomposition seemingly more convenient for that purpose, e.g., because it features the orientational symmetry better or because the decoration rule is simpler or whatever criterion of 'convenience' one is in favour of. Here, we are concerned with the question which patterns are *in principle* equivalent with respect to their ability of describing a certain long range translational order. Recognizing a spatial structure to be a decoration of another one simply is giving a rule how one can reconstruct every part of the former using just the information how the latter looks like in the surrounding. Hence, one is led to the conclusion that patterns are equivalent in this regard if and only if there is the possibility of a local transformation in both directions as in the examples above. This way the concept of local derivability arises which will now be described.

Let us now formulate our terms more precisely. We first introduce some basic notation. In the context here, by a *pattern* in a Euclidean vector space V we understand most generally a non-empty set of non-empty subsets of V. This definition includes tilings as well as sets of one-point sets, i.e., patterns of atomic positions. It would be possible to generalize the term 'pattern' even more allowing decorations with colours (or different types of atoms) etc, but this would only introduce some additional complications in notation and there is no difficulty at all to translate all the results given below to the more general case.

If $\mathcal{T}$ is a pattern in V and M a subset of V, we define $\mathcal{T} \sqcap M$ to be the subset of $\mathcal{T}$ that consists of all elements of $\mathcal{T}$ which intersect M. Furthermore, let B_r denote the closed ball around 0 in V with radius r.

Taking over a similar concept of Danzer [19] we call a pattern *weakly translationally repetitive* or (w, t)-repetitive for short, if for every radius $r > 0$ there is a radius $R > 0$ with the property that, for arbitrary points $x, x' \in V$, one can find a vector $t \in V$ such that $t + x \in x' + B_R$ and $\mathcal{T} \sqcap (x + B_r) = (-t + \mathcal{T}) \sqcap (x + B_r)$. These t's which are able to translate patches of the form $\mathcal{T} \sqcap (x + B_r)$, i.e., for which some $x \in V$ can be found such that $\mathcal{T} \sqcap (x + B_r) = (-t + \mathcal{T}) \sqcap (x + B_r)$, will be called *local displacement vectors* of $\mathcal{T}$ with respect to the radius r. Surely, repetitivity as defined above is not a sufficient condition for a structure to be 'diffractive', e.g., a point-like mass distribution on a (w, t)-repetitive set does not guarantee the Fourier transform to consist of sharp Bragg peaks—both the additional existence of other contributions like diffuse background or the total absence of Bragg peaks is possible. Generally, the problem of a characterization of the diffractive structures based on discrete tiling models 'in terms of x-space' is not solved at all. The term above is designed to fit the minimal conditions necessary for the development of our concept and it is not unlikely that quasiperiodic (discrete) tilings, that is to say tilings which have the property that Fourier transforms of distributions which respect these tilings consist of sharp Bragg peaks, are always (w, t)-repetitive.

Two patterns $\mathcal{T}$ and $\mathcal{T}'$ are *locally isomorphic* (or, synonymously, belong to the same LI-class) if for every $r > 0$ and every $x \in V$ there is some vector $t_1 \in V$ such that $(t_1 + \mathcal{T}) \sqcap (x + B_r) = \mathcal{T}' \sqcap (x + B_r)$ and some vector $t_2 \in V$ such that $(t_2 + \mathcal{T}') \sqcap (x + B_r) = \mathcal{T} \sqcap (x + B_r)$. (As in the definition of (w, t)-repetitivity here, for technical reasons, we operate with $\mathcal{T} \sqcap (x + B_r)$ rather than with the set of elements of $\mathcal{T}$ which are contained in $x + B_r$. If the elements of the patterns under consideration are bounded, as is the case for the usual tilings, this detail makes no difference at all.)

Periodic patterns form a special case of (w, t)-repetitive ones. If Λ is a lattice in V (discrete additive subgroup of V with $\mathrm{span}_{\mathbb{R}}(\Lambda) = V$), then a pattern $\mathcal{T}$ is Λ-periodic, if $t + \mathcal{T} = \mathcal{T}$ for every translation $t \in \Lambda$ and if Λ is the maximal lattice with this property. In the case of general (w, t)-repetitive patterns $\mathcal{T}$, one can define a $\mathbb{Z}$-module associated to $\mathcal{T}$ which reduces to the lattice in the periodic case. To do so, take the $\mathbb{Z}$-module $\Delta_r(\mathcal{T})$ generated by the possible local displacement vectors of the pattern $\mathcal{T}$ with respect to the radius r $(r > 0)$ as defined above. Obviously, $r > r'$ implies $\Delta_r(\mathcal{T}) \subseteq \Delta_{r'}(\mathcal{T})$. Therefore, it makes sense to define the limit module $\Delta(\mathcal{T})$ by $\Delta(\mathcal{T}) := \bigcap_{r>0} \Delta_r(\mathcal{T})$. The following statements can easily be demonstrated.

(1) If $\mathcal{T}$ is Λ-periodic and $\mathcal{T}'$ is locally is omorphic with $\mathcal{T}$, then $\mathcal{T} = s + \mathcal{T}'$ for some $s \in V$.

(2) If $\mathcal{T}$ is (w, t)-repetitive and $\mathcal{T}'$ locally isomorphic with $\mathcal{T}$, then $\mathcal{T}'$ is (w, t)-repetitive, and $\Delta(\mathcal{T}) = \Delta(\mathcal{T}')$.

Therefore, the module $\Delta(\mathcal{T})$ is an invariant of the LI class of $\mathcal{T}$. The meaning of this module in the case of patterns which are derived by a projection method will be described below.

Let us now come to the definition of the central concept of this section, namely that of 'local derivability'. The pattern $\mathcal{T}$ is said to be *locally derivable* from the pattern $\mathcal{T}'$ if there exists a radius $r>0$ such that $\mathcal{T}\sqcap\{x\}=(t+\mathcal{T})\sqcap\{x\}$ holds whenever $\mathcal{T}'\sqcap(x+B_r)=(t+\mathcal{T}')\sqcap(x+B_r)$. Clearly, this is a necessary and sufficient condition for the possibility to give a (formal) rule of how one has to construct the part of $\mathcal{T}$ surrounding a given point if one knows only the neighbourhood of that point in $\mathcal{T}'$ up to the radius r. Some properties of this relation are immediate.

(3) It is reflexive and transitive.

(4) If $\mathcal{T}_1$ is locally derivable from $\mathcal{T}_2$ and $\mathcal{T}_2'$ lies in the same LI-class as $\mathcal{T}_2$, then one can find $\mathcal{T}_1'$ in the LI-class of $\mathcal{T}_1$ which is locally derivable from $\mathcal{T}_2'$.

(5) If $\mathcal{T}$ is locally derivable from $\mathcal{T}'$ and $\mathcal{T}'$ is Λ'-periodic, then $\mathcal{T}$ is Λ-periodic with $\Lambda'\subseteq\Lambda$. On the other hand, if $\Lambda'\subseteq\Lambda$ and $\mathcal{T}$ is Λ-periodic, $\mathcal{T}'$ Λ'-periodic, then $\mathcal{T}$ is locally derivable from $\mathcal{T}'$.

(6) If $\mathcal{T}$ is locally derivable from $\mathcal{T}'$ and $\mathcal{T}'$ is (w, t)-repetitive, then $\mathcal{T}$ is (w, t)-repetitive and $\Delta(\mathcal{T}')\subseteq\Delta(\mathcal{T})$.

Statement (4) can be strengthened to 'simultaneous local derivability' of one LI class from another, provided that one element of the first LI class is locally derivable from one of the second. More precisely, if $\mathcal{T}_1$ is locally derivable from $\mathcal{T}_1'$, then one can find a radius r and an injective mapping $\mathcal{T}'\mapsto\mathcal{T}$ of the LI class of $\mathcal{T}_1'$ into the LI class of $\mathcal{T}_1$, such that for every x, $t\in V$ and $\mathcal{T}_2'$, $\mathcal{T}_3'$ locally isomorphic to $\mathcal{T}_1'$ the equation $\mathcal{T}_2'\sqcap(x+b_r)=(t+\mathcal{T}_3')\sqcap(x+B_r)$ implies $\mathcal{T}_2\sqcap\{x\}=(t+\mathcal{T}_3)\sqcap\{x\}$. In the case that $\mathcal{T}_1$ and $\mathcal{T}_1'$ are *mutually locally derivable* (m.l.d.) from each other this mapping turns out to be onto, and, furthermore, one can find a radius r' such that $\mathcal{T}_2\sqcap(x+B_{r'})=(t+\mathcal{T}_3)\sqcap(x+B_{r'})$ always implies $\mathcal{T}_2'\sqcap\{x\}=(t+\mathcal{T}_3')\sqcap\{x\}$.

This way, we have *established an equivalence relation between LI classes of patterns*. We may collect all elements of the LI classes of an equivalence class with respect to this relation into new classes called MLD classes. That is to say, the patterns $\mathcal{T}_1$ and $\mathcal{T}_2$ belong to the same MLD-class, if there is some $\mathcal{T}_2'$ locally isomorphic with $\mathcal{T}_2$ such that $\mathcal{T}_1$ and $\mathcal{T}_2'$ are mutually locally derivable (MLD) from each other. It is important to keep in mind that, by this definition, two patterns may belong to the same MLD class without being MLD; MLD classes are always unions of whole LI classes. As a consequence of 6), $\Delta(\mathcal{T})$ is an invariant even of the MLD class of $\mathcal{T}$. This gives the possibility for a first check whether at all two different patterns can belong to the same MLD class.

We have explained how the elements of two LI classes which are subsets of the same MLD class can be set into one-to-one correspondence. However, this one-to-one correspondence is not uniquely (or naturally) determined. For example, it is possible that two patterns $\mathcal{T}$ and $\mathcal{T}'$ are MLD whereas $\mathcal{T}'$ has some global orientational symmetry and $\mathcal{T}$ has not. Clearly, in such a case, there are several nontrivially different possibilities to provide such a one-to-one correspondence between the corresponding LI classes. In order to get rid of this sort of ambiguity, one has to consider a refined equivalence concept taking into account orientational symmetry. Namely, one defines the local derivability of a pattern $\mathcal{T}$ from $\mathcal{T}'$ to be symmetry preserving, if there is a radius r such that for every $x\in V$ and every Euclidean motion S of V the equation $\mathcal{T}'\sqcap(x+B_r)=(S\mathcal{T}')\sqcap(x+B_r)$ implies $\mathcal{T}\sqcap\{x\}=(S\mathcal{T})\sqcap\{x\}$. The corresponding equivalence relation between LI classes is then defined in an analogous way yielding S-MLD classes. Obviously, this relation respects generalized point symmetry. Furthermore, the occurrence of arbitrarily large patches with a certain symmetry in one pattern of an S-MLD class forces the occurrence of arbitrarily large patches with that symmetry in every element of this class.

It is now clear from the above considerations that $\mathcal{T}_{A_4^R}$ and Penrose tilings (scaled and oriented in the right way) belong to the same MLD-class (and even the same S-MLD class). Generally, in order to prove that two given patterns are MLD, it suffices to write down a local prescription how to pass from one pattern to the other and vice versa, as has been done in the example above. The task to show that given patterns do not belong to the same MLD-class (in no relative orientation and scaling) is somewhat more difficult. Let us describe in more detail which criteria for local derivability can be found in the case of patterns which are generated by a projection method, i.e., where one knows some embedding into a higher dimensional periodic structure.

In order to be able to formulate rigorous results we have to set up the framework precisely. Firstly, we will define a formalism which generates patterns from higher dimensional periodic structures in a way which, to our opinion, applies to most (if not all) projection methods that have been proposed in the literature for the derivation of quasiperiodic patterns. We will investigate to what extent this formalism is determined by the LI class of the generated patterns†. Secondly, we will tackle the question how the relation of local derivability can be formulated in terms of the projection method.

To start with the first point, let the 'physical' space V be embedded in a higher dimensional space $V_{\text{hyp}} = V \oplus V_{\text{int}}$, where V_{int} is referred to as 'internal' space (here we avoid the term 'perpendicular space', because there is no naturally given metric in hyperspace and we do not need one); let π, π_{int} denote the projections of V_{hyp} onto V, V_{int}, respectively. One has a mapping w which assigns to each subset P of V a subset $w(P)$ of V_{int}, the 'window' or 'existence domain' of P. We call *w a window mapping*, if the following conditions are fulfilled.

W1 $w(\emptyset) = \emptyset$; at least one $w(P)$ is non-empty; if $w(P) \neq \emptyset$ then P is bounded.

W2 For every $P \subseteq V$ we have $\overline{w(P)^0} = w(P)$ (the windows are the closures of their interiors (relative V_{int})).

W3 $\mathcal{K}_w := \{P + w(P) \mid P \subseteq V, w(P) \neq \emptyset\}$ is locally finite.

W4 $\Lambda_w := \{t \in V_{\text{hyp}} \mid t + \mathcal{K}_w = \mathcal{K}_w\}$ is a lattice in V_{hyp}, and its projection $\pi_{\text{int}}(\Lambda_w)$ is dense in V_{int}.

As a consequence of W2 and W3 the set $C_w := V_{\text{int}} \setminus \bigcup \{\partial w(P) \mid P \subseteq V\}$ of regular points of V_{int} with respect to w is dense in V_{int}. If $\pi_{\text{int}}(\Lambda_w)$ fails to be dense in V_{int}, then it is always possible to lower the dimension of V_{int} in order to embed the LI class of patterns which we are going to construct from w. An example for such a window mapping is the well known method of describing positions of atoms in a quasicrystal by so-called atomic hypersurfaces distributed periodically in a higher-dimensional space, provided these hypersurfaces are flat and parallel.

Now, for each regular point $c \in C_w$, a locally finite pattern in V is given by

$$\mathcal{T}_c^w := \{P \subseteq V \mid c \in w(P)\}.$$

One can show that, given $r > 0$, it is possible to find $R > 0$ such that for each $c_1, c_2 \in C_w$, $x_1, x_2 \in V$ there exists some $t \in \pi(\Lambda_w)$ which fulfils $t + x_1 \in x_2 + B_R$ and $\mathcal{T}_{c_1}^w \sqcap (x_1 + B_r) = (-t + \mathcal{T}_{c_2}^w) \sqcap (x_1 + B_r)$. *A fortiori*, the collection $\{\mathcal{T}_c^w \mid c \in C_w\}$ is contained in *one* LI class of (w, t)-repetitive patterns. Furthermore, it is almost immediate that for every $r > 0$,

† The authors are indebted to one of the referees for insisting on a more precise development of this point.

the projection $\pi(\Lambda_w)$ is contained in the module $\Delta_r(\mathcal{T}_c^w)$ ($c \in C_w$). Even more, one can find an $R>0$ such that $\Delta_R(\mathcal{T}_c^w) \subseteq \pi(\Lambda_w)$; let R_w be the minimal $R \geqslant 0$ for which this is true. (This is not that obvious in the general case but can be seen to be true by a more detailed inspection.) This means that from a certain size on patches can be displaced in a pattern $\mathcal{T}_c^w$ at most by elements of $\pi(\Lambda_w)$; for this reason it is always possible to distinguish firmly between 'tiles' from different orbits with respect to translations by elements of $\pi(\Lambda_w)$ in a completely local fashion. As a consequence, $\Delta(\mathcal{T}_c^w) = \Delta_{R_w}(\mathcal{T}_c^w) = \pi(\Lambda_w)$. The actual value of R_w depends sensitively on the very details of the window mapping w. The fact that for patterns obtained by a projection method as described above $\Delta(\mathcal{T}_c^w)$ can be determined without carrying out the limit procedure used in its definition may serve as a necessary criterion for the possibility that a given pattern can be generated by projection.

Let us consider the question to what extent the LI class of patterns $\mathcal{T}_c^w$ determines the window mapping w. Assume that we have two window mappings w_1 and w_2 which give rise to the same LI class; we assume further that both internal spaces have the same dimension (and can be identified therefore) and contain no nontrivial element of the lattices Λ_{w_1}, Λ_{w_2}. (We strongly suggest that the last two assumptions are superfluous, the first rather being a consequence of the very identity of the two LI classes; unfortunately, we have found no rigorous proof of this so far.) After some global shift in V one has $\bigcup\{\mathcal{T}_c^{w_1} \mid c \in C_{w_1}\} = \bigcup\{\mathcal{T}_c^{w_2} \mid c \in C_{w_2}\}$. Because $\Delta(\mathcal{T}_{c_1}^{w_1}) = \Delta(\mathcal{T}_{c_2}^{w_2})$ for some $c_i \in C_{w_i}$, the projections of Λ_{w_i} into $V_{\rm int}$ have to coincide ($i=1,2$). Therefore, it is possible to find a linear isomorphism L of $V_{\rm hyp}$ mapping Λ_{w_1} onto Λ_{w_2} and $V_{\rm int}$ onto $V_{\rm int}$. L can be decomposed into two parts, $L = L' \circ L_{\rm int}$, where $L_{\rm int}$ leaves $V_{\rm int}$ invariant and reduces to the identity on V, whereas L' is the identity on $V_{\rm int}$ but may shear the lattice Λ_{w_1} relatively to V. There is an obvious technique to reconstruct the windows $w_i(P)$ up to translations from the patterns $\mathcal{T}_{c_i}^{w_i}$ by lifting the elements $\pi(t) \in \pi(\Lambda_{w_i})$ which fulfil $\pi(t) + P \in \mathcal{T}_{c_i}^{w_i}$ up to their preimages $t \in \Lambda_{w_i}$ and projecting them down to $V_{\rm int}$. One sees at once that, if the decomposition $L = L' \circ L_{\rm int}$ cannot be arranged such that $L' = \mathbb{1}$, not both, $w_1(P)$ and $w_2(P)$, can be bounded and nonempty. But, as a consequence of W3, windows have to be bounded (except for the case that $\Lambda_{w_i} \cap V_{\rm int} \neq \{0\}$). So we may assume that $L' = \mathbb{1}$. If one passes from w_1 to w_1' defined by setting $w_1'(P) := L_{\rm int}(w_1(P))$, it is clear that w_1' produces the same set of patterns as w_1, and now one has $\Lambda_{w_1'} = \Lambda_{w_2}$. Next, one shows that for each $P \subseteq V$ there must be some translation $s \in V_{\rm int}$ such that $w_2(P) = w_1'(s+P)$ (otherwise one can construct a patch in $\mathcal{T}_c^{w_1'}$ no translate of which can occur in some $\mathcal{T}_c^{w_2}$, or vice versa), and with the same technique, one obtains that s has to be the same for all $P \subseteq V$. One is led to the conclusion that $\mathcal{T}_{c_1}^{w_1}$ is locally isomorphic to $\mathcal{T}_{c_2}^{w_2}$ for some $c_i \in C_{w_i}$ ($i=1,2$) if and only if there is a bijective affine transformation A of $V_{\rm int}$ and a translation vector $t \in V$ such that $w_2(P) = A(w_1(t+P))$ for all $P \subseteq V$.

So far we have dealt exclusively with the generation of one LI class of patterns. Our aim now is a characterization of local derivability in terms of properties of the window mappings themselves. We fix some window mapping w. To give the result in a comprehensive form, we introduce some convenient notation. If M, M_1, $M_2 \in V_{\rm int}$, let $M_1 \wedge M_2 := \overline{(M_1 \cap M_2)^0}$, $M_1 \vee M_2 := \overline{M_1 \cup M_2}$, and $\neg M := \overline{V_{\rm int} \backslash M}$. Let $\mathcal{A}$ be the set of all subsets M of $V_{\rm int}$ fulfilling $\overline{M^0} = M$. One verifies at once that $(\mathcal{A}, \wedge, \vee, \neg)$ is a Boolean algebra. Let $\mathcal{A}(w)$ denote the subalgebra of $\mathcal{A}$ generated by the set $\{w(P) \mid P \subseteq V\}$. The notation using logical symbols is chosen in view of the fact that, if P, P_1, $P_2 \subseteq V$, $c \in C_w$, then $c \in w(P_1) \wedge w(P_2)$, $c \in w(P_1) \vee w(P_2)$, $c \in \neg w(P)$, respectively, if and only if $P_1 \in \mathcal{T}_c^w$ and $P_2 \in \mathcal{T}_c^w$, $P_1 \in \mathcal{T}_c^w$ or $P_2 \in \mathcal{T}_c^w$, not $P \in \mathcal{T}_c^w$, respectively.

As outlined in the appendix, one arrives at the following result. Let w_1, w_2 be two window mappings with the same internal space $V_{\rm int}$; let $\Lambda_{w_1} \cap V_{\rm int} = \{0\}$. Then the following two statements are equivalent.

(E1) There are $c_1 \in C_{w_1}$, $c_2 \in C_{w_2}$ such that $\mathcal{T}_{c_2}^{w_2}$ is locally derivable from $\mathcal{T}_{c_1}^{w_1}$.

(E2) There is a bijective affine transformation A of $V_{\rm int}$ such that for w_2', defined by $w_2'(P) := A(w_2(P))$, $\Lambda_{w_2'} \supseteq \Lambda_{w_1}$ and $\mathcal{A}(w_2') \subseteq \mathcal{A}(w_1)$ hold.

The characterization of MLD classes now reads

> The window mappings w_1, w_2 with the same internal space $V_{\rm int}$ and $\Lambda_{w_1} \cap V_{\rm int} = \{0\}$ define the same MLD class of patterns if and only if there is an bijective affine transformation A of $V_{\rm int}$ such that for w_2', defined by $w_2'(P) := A(w_2(P))$, $\Lambda_{w_2'} = \Lambda_{w_1}$ and $\mathcal{A}(w_2') = \mathcal{A}(w_1)$ hold†.

Note that in (2) the linear part of A is completely fixed by the condition $\Lambda_{w_2'} \supseteq \Lambda_{w_1}$, a consequence of the projections of the lattices to be dense in the internal spaces.

An inspection of the definitions of the four LI-classes of patterns under consideration now leads to the following findings. Penrose and $\mathcal{T}_{A_4^{\rm R}}$ tilings fall into the same MLD-class as has been obtained at an earlier stage in a combinatorial fashion, $\mathcal{T}^*_{A_4}$ and $\mathcal{T}^*_{A_4^{\rm R}}$ define new MLD-classes each. (It is easy to see that, e.g., Penrose tilings and $\mathcal{T}^*_{A_4}$ tilings cannot fall into the same S-MLD class because there are arbitrarily large patches with fivefold symmetry in every Penrose tiling which is not true for $\mathcal{T}^*_{A_4}$ tilings‡.) It should be mentioned that it is possible to derive the Penrose tiling locally from the $\mathcal{T}^*_{A_4}$ tiling (in some convenient orientation and scaling), but not vice versa. We have strong evidence, furthermore, that the $\mathcal{T}^*_{A_4}$ tiling cannot be derived locally from any generalized Penrose tiling [13] at all. (The latter can be described in a 4D frame as well, wherefore an application of the technique mentioned above is possible.)

Another aspect of the concept introduced above is its application to deflation/inflation symmetries. A remarkable feature of various tilings found so far is that they may be rescaled by suitable factors without leaving the MLD-class, namely by the so-called deflation/inflation transformations. In this context one is guided to the following definition. Let $\lambda > 1$ $(0 < \lambda < 1)$, R be an orthogonal transformation in V. Then, λR is an inflation (deflation) transformation of the pattern $\mathcal{T}$, if there is a pattern $\mathcal{T}'$ such that $\lambda R(\mathcal{T})$ is locally isomorphic with $\mathcal{T}'$ and $\mathcal{T}$, $\mathcal{T}'$ are MLD. One observes at once that the usual inflation transformation by dilation by τ is an inflation transformation of the Penrose tiling and the tiling $\mathcal{T}^*_{A_4}$ in precisely that sense. On the other hand, it turns out that the tiling $\mathcal{T}^*_{A_4^{\rm R}}$ possesses no inflation transformation with $\lambda = \tau$, although it stems from the same 4D lattice as $\mathcal{T}_{A_4^{\rm R}}$ (the latter, belonging the same MLD-class as the Penrose tiling, is, of course, inflationable by τ). One can show that τ^4 is the smallest $\lambda > 1$ providing an inflation of this tiling. This example demonstrates that, in the case of a higher-dimensional embedding, not only the high-dimensional lattice but also more detailed features of the construction of the actual pattern are significant.

4. Conclusion

Motivated by the analysis of several quasiperiodic tilings with tenfold symmetry—two of which have been described in section 2—and their relation to each other we have proposed a general condition, namely 'mutual local derivability', under which different

† This statement is a generalization of an argument used by A Katz in the investigation of the possibility of local inflation rules (unpublished).

‡ The authors would like to thank R Penrose for an interesting discussion on this point.

tilings should be considered to be equivalent with respect to long range translational order. This concept makes no reference to an embedding and results in an arrangement of tilings in MLD-classes that have been discussed in an illustrative fashion without any attempt of a classification. Let us, at this point, reflect on the meaning of MLD-classes in the context of quasicrystallography, where, to our opinion, they should prove useful.

At present, there are quite a few attempts to reconstruct the microscopic arrangement of quasicrystalline materials by means of tiling models, although, despite some positive hints, no present experiment is able to convincingly prove or disprove the existence of physical objects which fit into (w, t)-repetitive tiling models. As we have argued, a tiling that describes the structure of a certain quasicrystal should belong to the same MLD-class as, e.g., the arrangement of atomic positions; it would not be unreasonable to demand that it even belongs to the same S-MLD class. If a tiling fulfils this condition it is not necessary that congruent tiles are decorated in the same way by the atoms. But, of course, it is always possible to construct a tiling from the same MLD-class such that even this is the case.

As follows from our above consideration, periodic structures belong to the same MLD-class if and only if their lattices are identical. In a way, MLD-classes are the (abstract) objects playing the role of lattices in the more general case of (w, t)-repetitive structures. It would then be the task to classify these objects in analogy to the Bravais classification of point lattices and to investigate the relation of MLD and S-MLD classes, which, in the periodic case, reduces to the investigation of space groups. For example MLD-classes which are connected by similarity transformations will be comprehended in the same class (as we have tacitly assumed when we emphasized that $\mathcal{T}^*_{A_4}$ and Penrose tiling are not mutually locally derivable 'in any relative orientation and scaling'). But, in the case of general MLD-classes, an investigation of the maximal generalized point symmetry of an MLD-class will not suffice for a satisfactory classification, which should result in a discrete set of classes. In the case of fivefold symmetry, e.g., it is possible to construct a continuous variety of MLD-classes (the generalized Penrose tilings) which all allow the same maximal symmetry [13]. On the other hand, it seems reasonable to demand such patterns as Penrose and $\mathcal{T}^*_{A_4}$ tilings, which possess both local deflation and matching rules, to be distinguished by a classification. That is why we think that a suitable classification scheme for quasiperiodic structures should be less coarse than the classification of Fourier modules alone.

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Appendix

We will present the argument leading to the equivalence of E1 and E2 in section 3.

If $x \in V$, $r > 0$, $c \in C_w$, then $\mathcal{T}^w_c \sqcap (x + B_r)$ is finite, i.e., $= \{P_1, \ldots, P_m\}$ (depending on x, r, c), say. Assume r_0 to be large enough such that $\mathcal{T}^w_c \sqcap (x + B_r) \neq \emptyset$ for all $x \in V$,

$r \geqslant r_0$, and $c \in C_w$. There are only finitely many $P \subseteq V$ such that $w(P) \cap w(P_1) \neq \emptyset$ and $P \cap (x + B_r) \neq \emptyset$, let them be $P'_1, \ldots, P'_{m'}$. If one now defines

$$p_w(x, r, c) := (w(P_1) \wedge \ldots \wedge w(P_m) \wedge \neg w(P'_1) \wedge \ldots \wedge \neg w(P'_{m'}))$$

then $p_w(x, r, c) \in \mathscr{A}(w)$, and one calculates easily that for $c' \in C_w$ we have $c' \in p_w(x, r, c) \Leftrightarrow p_w(x, r, c) = p_w(x, r, c') \Leftrightarrow \mathscr{T}^w_c \sqcap (x + B_r) = \mathscr{T}^w_{c'} \sqcap (x + B_r)$; furthermore, if $t \in \Lambda_w$, then $p_w(\pi(t) + x, r, \pi_{\mathrm{int}}(t) + c) = \pi_{\mathrm{int}}(t) + p_w(x, r, c)$ (and, if $t \in \Lambda_w \cap V_{\mathrm{int}}$, then even $p_w(x, r, \pi_{\mathrm{int}}(t) + c) = p_w(x, r, c)$).

Let now $\mathscr{T}$ be a locally finite pattern in V with bounded elements which is locally derivable from $\mathscr{T}^w_{c_0}$ for some $c_0 \in C_w$. Let $R > r_0$ be such that, for every x, $t \in V$, $\mathscr{T}^w_{c_0} \sqcap (x + B_R) = (t + \mathscr{T}^w_{c_0}) \sqcap (x + B_R)$ implies $\mathscr{T} \sqcap \{x\} = (t + \mathscr{T}) \sqcap \{x\}$. Choose from each $P \in \mathscr{T}$ some $x_P \in P$ such that, if $P \in \mathscr{T}$ and $s + P \in \mathscr{T}$, then $x_{s+P} = s + x_P$. Because $\mathscr{T}^w_{c_0}$ is (w, t)-repetitive, there is some $R' > 0$ such that $\mathscr{T}^w_{c_0} \sqcap (B_R + B_{R'})$ contains, up to translations by elements of $\pi(\Lambda_w)$, all possible patches $\mathscr{T}^w_{c_0} \sqcap (x + B_R)$. For that, $\mathscr{T} \sqcap B_{R'}$ contains, up to translations by elements of $\pi(\Lambda_w)$, all possible $P \in \mathscr{T}$. As $\mathscr{T}$ is locally finite, one concludes that $\{p_w(x_P, R, c_0 - \pi_{\mathrm{int}}(t)) \mid t \in \Lambda_w, \pi(t) + P \in \mathscr{T} \sqcap B_{R'}\}$ is finite, say $= \{W_1, \ldots, W_k\}$. One defines

$$w'(P) := (W_1 \vee \ldots \vee W_m)$$

and verifies that this way a window mapping is generated such that $C_{w'} \supseteq C_w$, $\Lambda_{w'} \supseteq \Lambda_w$, and $\mathscr{A}(w') \subseteq \mathscr{A}(w)$. This window mapping is designed such that $\mathscr{T} = \mathscr{T}^{w'}_{c_0}$ comes out and, furthermore, all $\mathscr{T}^{w'}_c$ are 'simultnaeously' locally derivable from the corresponding $\mathscr{T}^w_c$ $(c \in C_w)$: $\mathscr{T}^w_{c_1} \sqcap (x + B_R) = (t + \mathscr{T}^w_{c_2}) \sqcap (x + B_R)$ implies $\mathscr{T}^{w'}_{c_1} \sqcap \{x\} = (t + \mathscr{T}^{w'}_{c_2}) \sqcap \{x\}$ for all x, $t \in V$, c_1, $c_2 \in C_w$.

On the other hand, let w, w' be two window mappings (with the same internal space) such that $\Lambda_{w'} \supseteq \Lambda_w$, and $\mathscr{A}(w') \subseteq \mathscr{A}(w)$ (then, a fortiori $C_{w'} \supseteq C_w$). Let $P_1, \ldots, P_n \subseteq V$ be such that every element of $\bigcup \{\mathscr{T}^{w'}_c \mid c \in C_{w'}\}$ is a translate of some P_i by an element of $\pi(\Lambda_w)$. Each $w'(p_i)$ can be expressed applying a combination of the operations $\wedge$, $\vee$, and $\neg$ on a finite set $\{P_{i1}, \ldots, P_{in_i}\}$. As indicated above, the resulting term can be translated directly into a propositional function $\beta_i(c, Q_1, \ldots, Q_{n_i})$ built up by elementary expressions of the form $Q_j \in \mathscr{T}^w_c$. Then, for every $c \in C_w$, $t \in \pi(\Lambda_w)$, $i \in \{1, \ldots, n\}$ the statement $t + P_i \in \mathscr{T}^{w'}_c \Leftrightarrow \beta_i(c, t + P_{i1}, \ldots, t + P_{in_i})$ holds, a consequence of $\Lambda_{w'} \supseteq \Lambda_w$. Let $R \geqslant R_w$ (i.e. $\Delta_R(\mathscr{T}^w_c) = \pi(\Lambda_w)$) be such that for every $i \in \{1, \ldots, n\}$, $x \in P_i$, $j \in \{1, \ldots, n_i\}$ one has $P_{ij} \cap (x + B_R) \neq \emptyset$. Let $\tilde{\beta}_i(x, c, Q_1, \ldots, Q_{n_i})$ be the propositional function obtained from β_i by replacing every elementary expression $Q_j \in \mathscr{T}^w_c$ by $Q_j \in \mathscr{T}^w_c \sqcap (x + B_R)$. Then, $t + P_i \in \mathscr{T}^{w'}_c \sqcap \{x\} \Leftrightarrow \tilde{\beta}_i(x, c, t + P_{i1}, \ldots, t + P_{in_i})$ for all $t \in \pi(\Lambda_w)$ and $i \in \{1, \ldots, n\}$. Now, if $\mathscr{T}^w_c \sqcap (x + B_R) = (t + \mathscr{T}^w_c) \sqcap (x + B_R)$, then $t \in \pi(\Lambda_w)$, and one can conclude that $\tilde{\beta}_i(x, c, t' + P_{i1}, \ldots, t' + P_{in_i}) \Leftrightarrow \tilde{\beta}_i(x - t, c, -t + t' + P_{i1}, \ldots, -t + t' + P_{in_i})$ for every $t' \in \pi(\Lambda_w)$ and every $i \in \{1, \ldots, n\}$. As a consequence, $\mathscr{T}^{w'}_c \sqcap \{x\} = (t + \mathscr{T}^{w'}_c) \sqcap \{x\}$. For that, $\mathscr{T}^{w'}_c$ is locally derivable from $\mathscr{T}^w_c$. The equivalence of E1 and E2 now follows from the considerations concerning the uniqueness of window mappings (up to affine transformations) in section 3.

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